MDM4U – Sample Test 2 – Counting Methods – October 10, 2023

Name: Solutions

Knowledge	Application	Communication	Thinking	Total	Percent
27	22	16	25	90	%

Knowledge

1. Identify the formula for the following: /4

(a)	n!	$n(n-1)!$	(b)	n things arranged in a circle	$(n-1)!$
(c)	P(n, r)	$\frac{n!}{(n-r)!}$	(d)	C(n, r)	$\frac{n!}{(n-r)!\,r!}$

2. Calculate the following: /10

(a)	P(0,199)	error	(f)	3! x 4!	144
(b)	5!	120	(g)	C(9,3)	84
(c)	P(5, 5)	120	(h)	C(2,9)	error
(d)	P(51,2)	2550	(i)	P(809,0)	1
(e)	0!	1	(j)	C(678,1)	678

3. Consider the word "MAILBOX". Fill in the values indicated by the column headings. /10

		Equation, with factorials	Final Answer
(a)	Permutations with all letters.	$7!$	5040
(b)	Permutations with 4 letters.	${}_7P_4 = \frac{7!}{3!}$	840
(c)	Permutations; M in the front.	$6!$	720
(d)	Circular permutations.	$6!$	720
(e)	Combinations with 4 letters.	${}_7C_4 = \frac{7!}{4!\,3!}$	35

4. Identify the Excel formula for the following: /3

(a)	${}_9P_4$	=Permut(9,4)
(b)	${}_7C_5$	=combin(7,5)
(c)	12!	=fact(12)

Application

For each of these problems provide a full solution. Evaluate.

5. Consider the word "OCTOBER". Find out how many things result in each case. /8

(a) Permutations using all letters.

- 7 letters
- O's repeat

$= \frac{7!}{2!}$

$= 2,520$ ways

(b) Permutations of 3 letters, all the same.

- No letters have 3 same

$= 0$ ways

(c) Permutations of 3 letters where 2 letters the same, one different.

[0][0][5] ↓ E,T,B,C,R

$= 5$ letters $\times 3$ places

$= 15$ ways

(d) Permutations with all letters, OO at the end.

[5][4][3][2][1][0][0] fixed O's at end

$= 5!$

$= 120$ ways

6. A class has 12 right-handed people and 8 left-handed people. Four children are chosen to be in a play. What is the probability that there is an equal number of left and right-handed people chosen? /6

Left	Right	Total
n 8	n 12	n 20
r 2	r 2	r 4

$P(\text{equal num}) = \frac{n(\text{equal})}{n(\text{total})}$

$= \frac{\binom{8}{2}\binom{12}{2}}{\binom{20}{4}}$

$= \frac{28 \times 66}{4845}$

$= \frac{1848}{4845}$

$= \frac{616}{1615}$ or 0.3814

7. Ella Vader wants a good lock for her locker. Which lock is the most secure? Prove it to Ella using counting principles. /8

Oria Lock: Each dial has the digits 0-9 on it.

Positions

[10][10][10][10]

$= 10^4$

$= 10,000$

Master Lock: The same number can not be used back-to-back. The combination is 3 numbers each between 0 and 39.

Positions

[40][39][39]

$= 40 \times 39 \times 39$

$= 60,840$

Wordlock Lock: Each dial has 8 letters on it.

Positions

[8][8][8][8][8]

$= 8^5$

$= 32,768$

Ella should use the Master Lock because it has the highest number of possible combinations. A brute-force hacker would need to try more combinations (on average) than the other 2 locks.

Communication

8. Classify each type of problem using the appropriate letter in the last column. /10

A. Counting, **B.** Factorial, **C.** Permutations, **D.** Perms with Restrictions, **E.** Perms with Repeats, **F.** Circular Perms, **G.** Combinations

	Problem	Classification
(a)	The number of ways you can choose 5 kittens from a litter of 10 kittens.	G
(b)	You choose a group of 4 from a club of 200 to greet the Governor General.	G
(c)	The number of license plates in Ontario.	A
(d)	From a horse race of 18 horses, there is a first, second and third place finisher.	C
(e)	The number of rearrangements of TURKEY with T in the first position.	D
(f)	The ways to arrange 20 pearls around a necklace.	F
(g)	How many ways you can arrange all the letters in OWLS?	B
(h)	From a team of 18 students, you are picking a captain and assistant captain.	C
(i)	(Two types) The number of arrangements of SEALS with E in the first position.	D, E
(j)	(Two types) The ways to position 4 red, 6 yellow and 1 pink tulips around a flagpole.	E, F

9. When a guest complains to the wedding planner about their table's seating plan, the planner says, "I know it's not perfect, but it really is the only arrangement that worked." If twelve people are seated around a table, explain why that is unlikely. /2

In a circular arrangement, things can be positioned $(n-1)!$ ways, or $11! = 39,916,800$ ways in this instance. Even with restrictions to eliminate some of the possibilities, it is extremely unlikely the planner manually tried thousands of table arrangements to find this "one" working arrangement.

10. (a) Provide one way that permutation and factorial questions are similar, and one that they differ. /2

Same	- Both are used when order matters - A permutation is a factorial; it is simply a chopped off factorial.
Differ	- A permutation uses only some of the objects, factorials use all of the objects.

(b) Provide one way that permutation and combination questions are similar, and one that they differ. /2

Same	$P = \frac{n!}{(n-r)!}$ $C = \frac{n!}{r!(n-r)!}$ - Both are used to calculate a 'count' of something - Both have $n!$ in the numerator of formulas - Both have $(n-r)!$ in the denominator of formulas
Differ	- Permutations are used when order matters. Combinations are used when it doesn't matter. - Generally, permutations are larger than combinations. This is because BA + AB are both counted for permutations but are counted once for combinations (BA = AB in combins)

Thinking

11. Simplify. Do not evaluate. /6

(a) $\frac{89!}{90!} = \frac{89!}{90 \times 89!} = \frac{1}{90}$

(b) $\frac{\left(\frac{30 \times 29!}{42!}\right)}{\left(\frac{30!}{41!}\right)} = \frac{30!}{42!} \times \frac{41!}{30!} = \frac{1}{42 \times 41!} \times \frac{41!}{1} = \frac{1}{42}$

(c) $\frac{C(22,4)}{C(23,4)} = \frac{22!}{18!\,4!} \times \frac{4!\,19!}{23!} = \frac{22!}{18!} \times \frac{19 \times 18!}{23 \times 22!} = \frac{19}{18}$

12. How many arrangements of ALGORITHM begin with a vowel but do **not** end with a consonant? /4

Counts
- 3 (A O I) Vowels
- 6 (L, G, R, T, H, M) Consonants
- 9 letters total

[3] [7] [6] [5] [4] [3] [2] [1] [2]

Vowel: 3 to start. Other left over letters. Vowel: 2 left. (a vowel)

ways = $7! \times 3 \times 2$
= 30,240

13. How many 4 letter permutations of "APPELLEE" exist? (Hint: there are 4 cases) /10

Case 1: 3 the same, 1 Diff

E E E 3 (3: A, P, L)

$= \frac{4!}{3!} \times 3$ letters
$= 4 \times 3$
= 12 ways

Case 2: 2 same, 2 same

P P E E
L L P P
E E L L

$= \frac{4!}{2!\,2!} \times 3$ sets
$= 6 \times 3$
= 18 ways

Case 3: 2 same, 2 diff

→ One set e.g. P P A E (Arrange) $= \frac{4!}{2!} = 12$

→ One Pairs Possible Sets
P P $\binom{3}{2}$ letters, spots $= C(3,2) = 3$

→ Possible Pairs = PP, LL, EE

Ways = 3 pairs × 12 arrange × 3 sets
= 108 ways

Case 4: All Diff

4 3 2 1 (A, P, E, L) $= 4! = 24$

Total
= 12 + 18 + 108 + 24
= 162

14. Solve for n: $\frac{(n+1)!}{(n-1)!} = 20$ /3

$\frac{(n+1)(n)(n-1)!}{(n-1)!} = 20$

$(n+1)(n) = 20$

$n^2 + n = 20$

$n^2 + n - 20 = 0$

$(n-4)(n+5) = 0$

∴ n is either 4 or −5
−5 is probably inadmissible.