1.1 Exercises: Probability via Sample Space & Counting

1. List 3 probability statements, like those discussed in the introduction to this chapter, that you have found in magazines, newspapers, etc., or that you have heard on the radio or television.

2. A red die and a blue die are rolled and the number on the top face of each die is recorded. A sample space for this situation is

$$S = \{(1, 1), (1, 2), \cdots, (6, 6)\}$$

where, for example, (3, 4) indicates the outcome that the red die comes up 3 and the blue die comes up 4.

   a. List the outcomes in each event.

   A: the faces on the dice are identical
   B: the face on the red die is less than the face on the blue die
   C: the face on the blue die is less than 2 times the face on the red die
   D: neither face is greater than 4
   E: the total of the faces is 7

   b. Assign the probability $\frac{1}{36}$ to each outcome in S. Calculate $P(A)$, $P(B)$, $P(C)$, $P(D)$, and $P(E)$.

3. Two-digit numbers are generated by a computer program so that numbers between 00 and 99 occur with equal frequency in the long run but with no apparent pattern in the short run. A probability model for this situation has the sample space

$$S = \{00, 01, 02, 03, \cdots, 98, 99\}$$

with probability $\frac{1}{100}$ for each outcome.

   a. Describe each event in words.

   $A = \{00, 11, 22, \cdots, 99\}$

   notA

   $B = \{09, 18, 27, \cdots, 90\}$

   $C = \{51, 52, 53, \cdots, 68\}$

   $D = \{01, 11, 21, \cdots, 91\}$

   b. Find the probability of each event in part (a).

   c. Find the odds in favour of each event in part (a).

4. Four runners, Bruce, Harish, Michael, and Tyler, plan to compete in a marathon. Based on past performances, there is no information to predict that any runner is better than any other. Assuming that there are no ties, a probability model to describe the outcome of the race is

$$S = \{bhmt, bhtm, bmht, \cdots, tmhb\}$$

where, for example, *thmb* represents the outcome in which Tyler finishes first, Harish second, Michael third, and Bruce fourth. Each outcome in S is assigned the probability $\frac{1}{24}$.

a. List the outcomes in each event and find the probability of the event.

E: Bruce won the marathon
F: Tyler finished last
G: Harish and Tyler were the last two runners to cross the finish line
J: the runners finished in alphabetical order
K: Michael did not win the marathon

b. Describe in words the complement of each event in part (a).

c. Find the odds against each event in part (a).

5. Three coins are tossed simultaneously, and the showing face on each coin is recorded.

a. Define a sample space for this situation.

b. List the outcomes in each event.

F: no heads are obtained
G: at least 1 tail is obtained
H: the numbers of heads and tails differ by 1
J: the showing face on the second coin is a head
K: exactly 2 heads are obtained

c. Assign the probability $\frac{1}{8}$ to each outcome in the sample space S. Calculate $P(F)$, $P(G)$, $P(H)$, $P(J)$, and $P(K)$.

6. From a lottery mixing box containing 5 balls numbered 1, 2, 3, 4, and 5, 2 balls are selected, one after the other without replacement.

a. Define a sample space for this situation which keeps track of the 2 balls selected and the order of selection.

b. List the outcomes in each event.

L: the second ball has an even number
M: ball 1 is selected first
N: the largest number selected is 4
O: the numbers on the balls selected differ by 1
Q: the sum of the numbers drawn exceeds 5

c. Assign the probability $\frac{1}{20}$ to each outcome in S. Calculate $P(L)$, $P(M)$, $P(N)$, $P(O)$, and $P(Q)$.

7. The names of 2 students on the students' council are chosen by drawing names from a hat. The selected students will represent the school at a provincial convention. The students' council consists of 6 students whose names are Asbir, Emmy, Lukas, Rolf, Susan, and Zia.

   a. Define a sample space for this situation which records the 2 students chosen but not the order in which they are chosen (using initials can save space).

   b. List the outcomes in each event.

      R: Lukas is selected
      W: Emmy is selected, but Asbir is not selected
      T: Zia is not selected
      U: no 2 of Asbir, Zia, and Rolf are selected to attend the convention
      V: neither Rolf nor Lukas is selected

   c. Assign the probability $\frac{1}{15}$ to each outcome in S. Calculate $P(R)$, $P(W)$, $P(T)$, $P(U)$, and $P(V)$.

8. A probability model is defined with the sample space

$$S = \{1, 2, 3, 4\}$$

Suppose $P(1) = P(3)$, the odds in favour of an even outcome are 2:1, and $P(notA) = \frac{1}{3}$, where A is the event that an outcome is less than 3. Find the probability assigned to each outcome in S.

9. A sample space S contains 6 outcomes, e_1, e_2, e_3, e_4, e_5, and e_6. If the corresponding probabilities are p_1, p_2, p_3, p_4, p_5, and p_6 (all real numbers between 0 and 1), determine which conditions completely specify a probability model.

   a. $p_1 = p_2, p_3 = p_4, p_5 = p_6, p_1 + p_3 + p_5 = 0.4$

   b. $p_1 + p_2 + p_3 = p_4 + p_5 + p_6$

   c. $p_1 = p_4 = p_5 = p_6, p_2 + p_3 = 0.6$

   d. $p_1 = 2p_2 = 3p_3 = 4p_4 = 5p_5 = 6p_6$

   e. $p_1 = 2p_2, 3p_3 = 4p_4, 5p_5 = 6p_6$

10. A probability model that describes the condition of 3 electrical components after 10 000 h of use has the sample space

$$S = \{WWW, WWN, WNW, NWW, WNN, NWN, NNW, NNN\}$$

where, for example, the outcome WNW describes the case in which component 1 is working, component 2 is not working, and component 3 is working. Based on laboratory testing, the probabilities assigned to the outcomes in S are

$$P(WWW) = \frac{1}{27}$$
$$P(WWN) = P(WNW) = P(NWW) = \frac{2}{27}$$
$$P(WNN) = P(NWN) = P(NNW) = \frac{4}{27}$$
$$P(NNN) = \frac{8}{27}$$

   a. Show that these probabilities define a probability model.

   b. Find the probability of each event.

      A: component 2 is not working
      B: at least 1 component is working
      C: component 1 is working but component 3 is not

11. A fair coin is tossed until either it turns up heads or there have been 6 tosses. A probability model to describe this situation is

$$S = \{H, TH, TTH, TTTH, TTTTH, TTTTTH, TTTTTT\}$$

with probabilities $\left(\frac{1}{2}\right)^k$, where k is the number of tosses in the outcome.

a. Show that this is a probability model.

b. Find the probability of each event.

G: at least 2 tosses are made
H: an odd number of tosses are made
I: no heads occur

12. The following statement was recently observed on a package in a store.

"Inside 688 800 specially marked packages is a 0.3 carat zircon (approximate retail value \$15) in a sealed envelope. In 138 of these packages, randomly distributed across Canada, there is an extra notice that the holder is eligible to win a 0.3 carat natural diamond (approximate retail value \$350) by complying with the contest rules printed inside the envelope. The odds of receiving a natural diamond notice are 1 in 5000."

What is wrong with the statement as it appeared on the package?

Problems 6.1

1. Watto 3/49 is a new lottery. Participants choose 3 numbers between 1 and 49; the numbers need not be distinct. The winners are chosen by selecting 3 balls, with replacement, from a mixer containing balls numbered 1 to 49. A winning ticket must have at least 2 of the 3 numbers selected.

a. Describe a sample space that gives all possible selections of 3 numbers from the mixer. How many outcomes are there?

b. If all the outcomes are equally likely, calculate the probability that a ticket with numbers 1, 2, and 3 wins a prize.

c. Calculate the probability that a ticket with numbers 1, 1, and 1 wins a prize.

d. Calculate the odds against winning a prize if the numbers on the ticket are 1, 2, and 2.

2. Repeat parts (a) and (b) of Problem 1, assuming that the numbers on a ticket must be distinct.

3. A fair coin is tossed repeatedly until tails is observed.

a. Define a sample space for this situation.

b. Assign the probability $\left(\frac{1}{2}\right)^j$, where $j = 1, 2 \ldots$, to each outcome in the sample space consisting of j tosses. Calculate the probability of each event.

E: an even number of tosses is observed
O: the number of tosses is exactly divisible by 3
T: at least 10 tosses are observed

4. Consider the sample space $S = \{0, 1, 2, \ldots, n\}$ with probabilities

$$P(i) = \binom{n}{i} p^i (1 - p)^{n-i} \quad \text{for } i = 0, 1, \ldots, n$$

where $0 < p < 1$.

a. Use the binomial theorem to prove that these probabilities satisfy the requirements for a probability model.

b. Calculate the probability that the outcome is not 0.

c. If n is even, calculate the probability that the outcome is even.

Exercises 6.1

2. a. $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$
$B = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$
$C = \{(1, 1), (2, 3), (2, 2), (2, 1), (3, 5), (3, 4), (3, 3), (3, 2), (3, 1)$, and all pairs with red $\geq 4\}$
$D = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$
$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

b. $P(A) = \frac{1}{6}, P(B) = \frac{15}{36}, P(C) = \frac{3}{4}, P(D) = \frac{4}{9}, P(E) = \frac{1}{6}$

3. a. A: number is an integer multiple of 11, *notA*: number is not an integer multiple of 11, B: number has digits which sum to 9, C: number between 51 and 68 inclusive D: number ending in 1

b. $P(A) = \frac{1}{10}, P(notA) = \frac{9}{10}, P(B) = \frac{1}{10}, P(C) = \frac{9}{50}, P(D) = \frac{1}{10}$

c.

Event:	A	*notA*	B	C	D
Odds in favour:	1:9	9:1	1:9	9:41	1:9

4. a. $E = \{bhmt, bhtm, bmth, bmht, btmh, bthm\}$, $P(E) = \frac{1}{4}$; $F = \{bhmt, hbmt, bmht, mbht, hmbt, mhbt\}$, $P(F) = \frac{1}{4}$; $G = \{mbht, bmht, mbth, bmth\}$, $P(G) = \frac{1}{6}$; $J = \{bhmt\}$, $P(J) = \frac{1}{24}$; $K = not\{mbht, mbth, mtbh, mthb, mhtb, mhbt\}$, $P(K) = \frac{3}{4}$

b. *notE*: Bruce did not win the marathon; *notF*: Tyler did not place last; *notG*: Harish or Tyler or both finished first or second; *notJ*: the finishing order is not alphabetical; *notK*: Michael won the marathon.

c.

Event:	E	F	G	J	K
Odds against:	3:1	3:1	5:1	23:1	1:3

5. a. $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ **b.** $F = \{TTT\}$; $G = \{TTT, TTH, THT, HTT, HHT, HTH, THH\}$; $H = \{HHT, HTH, THH, HTT, THT, TTH\}$; $J = \{HHH, THH, HHT, THT\}$; $K = \{HHT, HTH, THH\}$ **c.** $P(F) = \frac{1}{8}, P(G) = \frac{7}{8}, P(H) = \frac{3}{4}, P(J) = \frac{1}{2}, P(K) = \frac{3}{8}$

6. a. $S = \{12, 13, 14, 15, 21, 23, 24, 25, 31, 32, 34, 35, 41, 42, 43, 45, 51, 52, 53, 54\}$ **b.** $L = \{12, 32, 42, 52, 14, 24, 34, 54\}$; $M = \{12, 13, 14, 15\}$; $N = \{14, 24, 34, 41, 42, 43\}$; $O = \{12, 21, 23, 32, 34, 43, 45, 54\}$; $Q = \{15, 51, 24, 42, 25, 52, 34, 43, 35, 53, 45, 54\}$ **c.** $P(L) = \frac{2}{5}, P(M) = \frac{1}{5}, P(N) = \frac{3}{10}, P(O) = \frac{2}{5}, P(Q) = \frac{3}{5}$

7. a. $S = \{AE, AL, AR, AS, AZ, EL, ER, ES, EZ, LR, LS, LZ, RS, RZ, SZ\}$ **b.** $R = \{LA, LE, LR, LS, LZ\}$; $W = \{EL, ER, ES, EZ\}$; $T = \{AE, AL, AR, AS, EL, ER, ES, LR, LS, RS\}$; $U = \{AE, AL, AS, EL, ER, ES, EZ, LR, LS, LZ, RS, SZ\}$; $V = \{AE, AS, AZ, ES, EZ, SZ\}$ **c.** $P(R) = \frac{1}{3}, P(S) = \frac{4}{15}, P(T) = \frac{2}{3}, P(U) = \frac{4}{5}, P(V) = \frac{2}{5}$

8. $P(1) = P(3) = P(4) = \frac{1}{6}, P(2) = \frac{1}{2}$

9. a. No **b.** No, e.g. $p_1 + p_2 + p_3 = \frac{3}{4} = p_4 + p_5 + p_6$ **c.** No, e.g. $p_1 = p_4 = p_5 = p_6 = 0.3$, $p_2 + p_3 = 0.6$ **d.** Yes **e.** No, $p_1 = 1.0, p_2 = 0.5, p_3 = 0.4, p_4 = 0.3, p_5 = 0.6, p_6 = 0.5$

10. b. $P(A) = \frac{2}{3}, P(B) = \frac{19}{27}, P(C) = \frac{2}{9}$

11. b. $P(G) = \frac{1}{2}, P(H) = \frac{21}{32}, P(I) = \frac{1}{64}$

12. Odds in favour of receiving prize are 138:688 662 $\neq$ 1:5000
1:4990.5